Advanced Geometry

Interactive textbook

Surfaces and curves of constant width

This interactive textbook is destined for students who don't like formulae, but experience great interest in images and their dynamic transformation. The chapter presents the topic "Surfaces and curves of constant width". Dynamic GInMA illustrations provide maximum interactivity, gives the opportunity to work with 3D-images and make it convenient to introduce the study in training courses on geometry.

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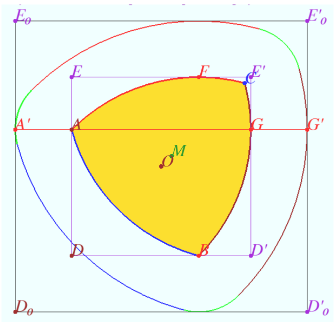

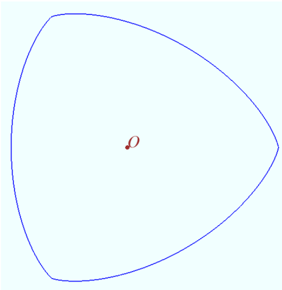

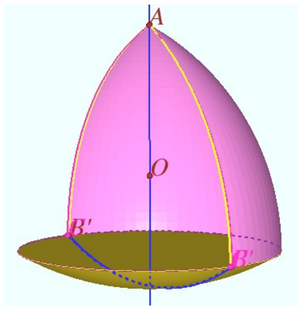

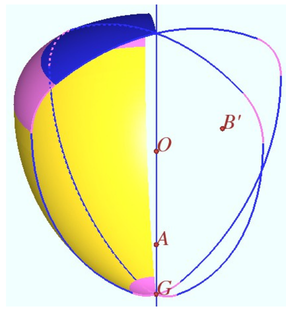

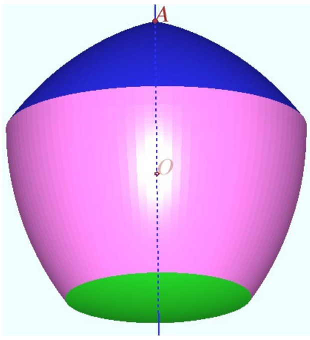

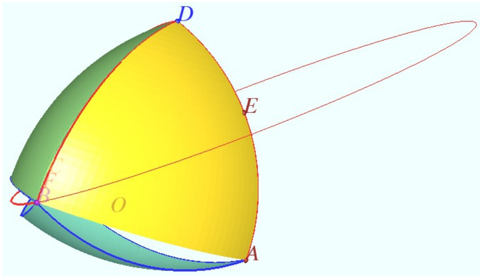

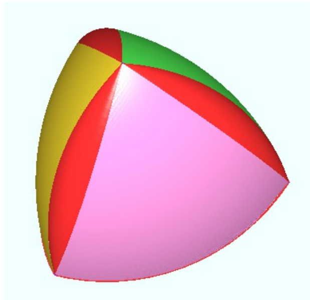

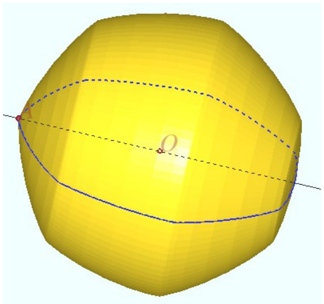

1 Curve of Constant Width

A curve of constant width is a planar closed convex curve, which width is the same value regardless of the orientation of the curve. The width of a closed convex curve is defined as the distance between parallel lines bounding it ("supporting lines").Supporting lines are tangent to the curve at the different sides. Curves of constant width have the same width regardless of their orientation between the parallel lines.

We may also define a curve of constant width as a planar closed convex curve, for which the length of the orthogonal projection on any straight line is equal to the same value.

A circle is a curve of constant width equal to its diameter. A curve of constant width ***h*** may be inscribed in a square with a side of ***h***, the orientation of which is arbitrary. In other words, the curve of constant width, when rotated in a square, make contact with all its four sides (it remains inscribed in the square).

1.1 Reuleaux triangle and its analogue

Reuleaux triangle: construction and study. Let us take an equilateral triangle with ***h*** side and every two vertices connect by an arc of radius ***h*** centered at the third vertex. The constructed curve is the Reuleaux triangle. In Figure 1 the triangle is determined by the center ***O*** and the vertex ***A***, which allows you to rotate the triangle. A square with ***h*** side is located in the plane ***AOD***. The square is determined by the point ***D***.

Check that at any position of the triangle it is inscribed in a square.

"Kid's" question. The triangle has three sides and three vertices. How it always manages to touch each of four sides of the square?

Analogue of the Reuleaux triangle. Let us take an equilateral triangle with ***a*** side, extend the sides beyond the vertices, attach equal arcs centered at the nearest vertex with the angle of 60° and ***b*** radius to the vertices between the extensions of sides. Then we connect the free ends of the arcs by the arcs centered at the opposite vertex with the angle of 60° and radius equal to ***a*** + ***b***. The constructed curve is the analogue of the Reuleaux triangle. The shape consists of three (green) arcs with the radius ***AA′*** and three arcs with the radius ***AA′*** + ***AG***. Figure 1 shows the analog of the Reuleaux triangle inscribed in the outer square, the size of which is determined by the point $\boldsymbol{D_0}$.

Check that the shape has the constant width equal to ***h* = *2b* + *a***. Choose the parameters ***a*** and ***b*** using the points *O, D*, and D_0. Move the shape using the point ***A.*** Try to understand why its width does not change when rotating.

Task 1. Construct a Reuleaux pentagon.

Task 2. Find ***n*** for which we can construct a Reuleaux ***n***-gon (a Reuleaux polygon).

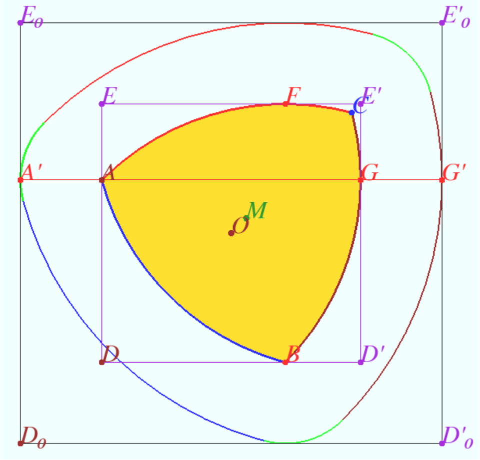

Figure 1. Reuleaux triangle and its analogue

In Figure 2 the points *O, A* and *D* determine the Reuleaux triangle analogue The part of pedal curve is constructed. The points *E* and *E*' determine the tangents. *Explore* the pedal curve.

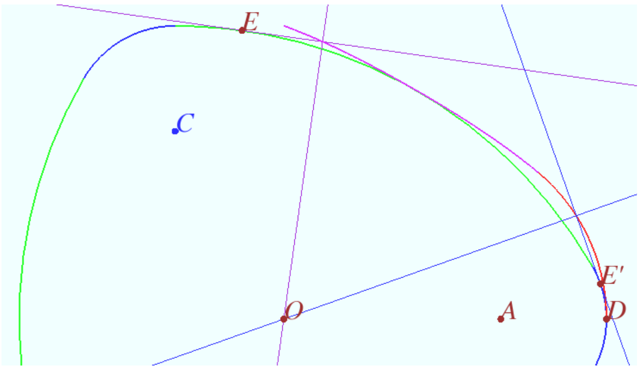

Figure 2. Reuleaux triangle analogue and its pedal curve

1.2 Algebraic curves of constant width

There are known [4] algebraic curves of constant width. These curves are constructed using properties of a pedal curve. The pedal curve of the given curve about some point (on the curve) is the curve containing the bases of the perpendiculars dropped from this point to the tangents of the given curve. Let the origin ***O*** is located inside the curve, the angle between the perpendicular dropped onto a tangent to the curve and the positive direction of the x-axis is equal to **φ**, the distance from ***O*** to the base ***H*** of the perpendicular dropped from ***O*** onto the tangent is equal to $p(\varphi)$. Then the equation of the tangent has the form

$$\frac{dy}{dx} = -ctg\,\varphi = \frac{y - p(\varphi)\sin\varphi}{x - p(\varphi)\cos\varphi}.$$

The curve equation has the form:

$$\begin{cases} x = p(\varphi)\cos\varphi - p'(\varphi)\sin\varphi, \\ y = p(\varphi)\sin\varphi + p'(\varphi)\cos\varphi. \end{cases}$$

A characteristic property of the constant width curve is expressed by the formula:

$$p(\varphi) + p(\varphi + \pi) = 2b = const.$$

The periodic solution $p(\varphi) = p(\varphi + 2k\pi)$ may be written in the form:

$$\begin{cases} p(\varphi) = a\cos(2n+1)\varphi + b,\ n \in N, \\ p'(\varphi) = -(2n+1)a\sin(2n+1)\varphi. \end{cases}$$

The solution in the general case is not convex.

In Fig.3 the algebraic curve of constant width with parameters ***n, a,*** and ***b*** is shown by blue. The pedal curve of point *B* is shown by pink. The point ***C*** position is determined by the parameter **θ**. *Explore* the pedal curve.

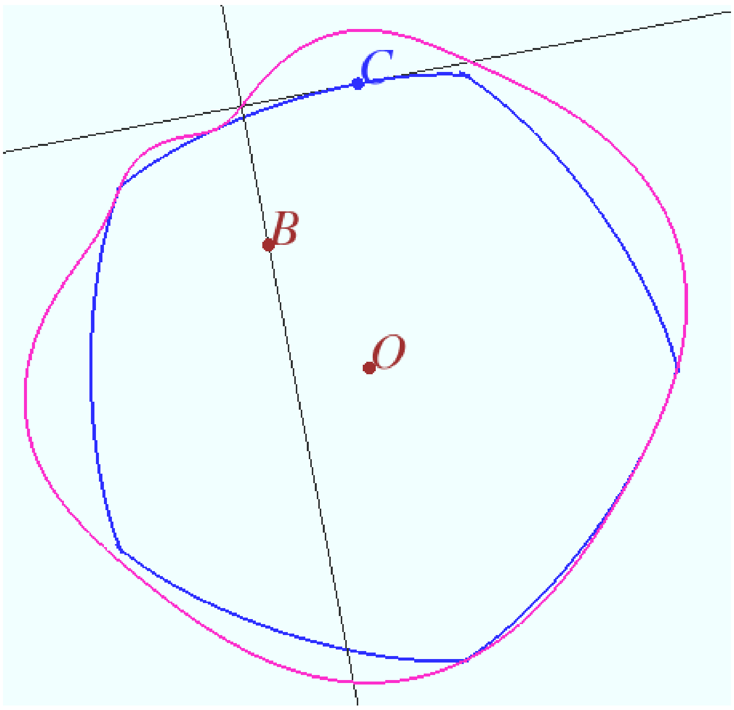

Figure 3. Algebraic curve of constant width and a pedal curve

2 Surface of constant width

A surface of constant width is a convex three-dimensional shape for which the width, measured by the distance between two opposite parallel planes touching its boundary, is the same regardless of the direction of those two parallel planes. One defines the width of the surface in a given direction to be the perpendicular distance between the parallels perpendicular to that direction.

A surface of constant width may be inscribed in a cube of the side ***h***, the orientation of which is arbitrary. In other words, the surface of constant width, when rotated in a cube, make contact with all its faces (it remains inscribed in the cube).

2.1 Surface of revolution of a Reuleaux triangle

The original curve is a Reuleaux triangle, obtained from the equilateral triangle with ***h*** side, in which every two vertices are connected by an arc of radius ***h*** centered at the third vertex. In Figure 3 the Reuleaux triangle is determined by the center ***O*** and the vertex ***A***.

The surface of revolution of a Reuleaux triangle we obtain by Reuleaux triangle rotating about one of its axis of symmetry, for example the symmetry axis ***OA***. The surface of revolution of a Reuleaux triangle through one of its symmetry axes forms a surface of constant width.

Figure 3 shows the half of the surface of revolution of a Reuleaux triangle.

The point ***B'*** controls the rotation.

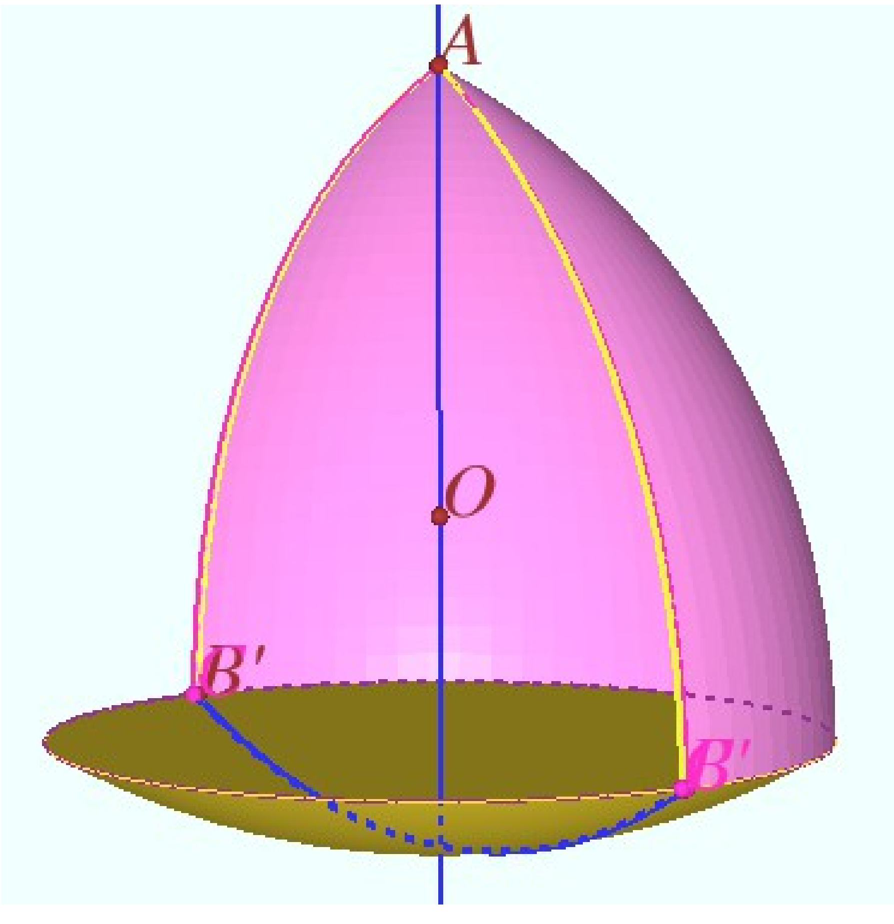

Figure 4. Half of the surface of revolution of a Reuleaux triangle

2.2 Surface of revolution of a Reuleaux triangle analogue

The original curve is the analogue of the Reuleaux triangle. In Figure 4 the Reuleaux triangle analogue consists of three (pink) arcs with the radius ***AG*** and three (blue) arcs with the radius ***AB* + *AG***. The shape is determined by the center ***O***, ***A*** point from the reference equilateral triangle and the vertex ***G***. If we bring together the points ***A*** and ***G***, the shape approaches to the Reuleaux triangle. If we bring together the points ***A*** and ***O***, it is approaching to the shape of a circle.

The surface of revolution of a Reuleaux triangle analogue we obtain by the Reuleaux triangle analogue rotating about one of its axis of symmetry, for example the symmetry axis ***OA.*** The surface of revolution of a Reuleaux triangle analogue through one of its symmetry axes forms a surface of constant width.

Figure 5 shows the third part of the surface of revolution of a Reuleaux triangle analogue.

The point ***B'*** controls the rotation.

Make sure that the solid has a constant width. *Study* may be performed with use of the cube ***DEF***. Set the angular position of the cube using the points ***E*** and ***F***. Then move the cube along the edges using the point ***D***. Examining the cube faces, we find the protruding elements of the solid as colored spots on the cube edges. Move the cube using the point ***D*** and delete these spots.

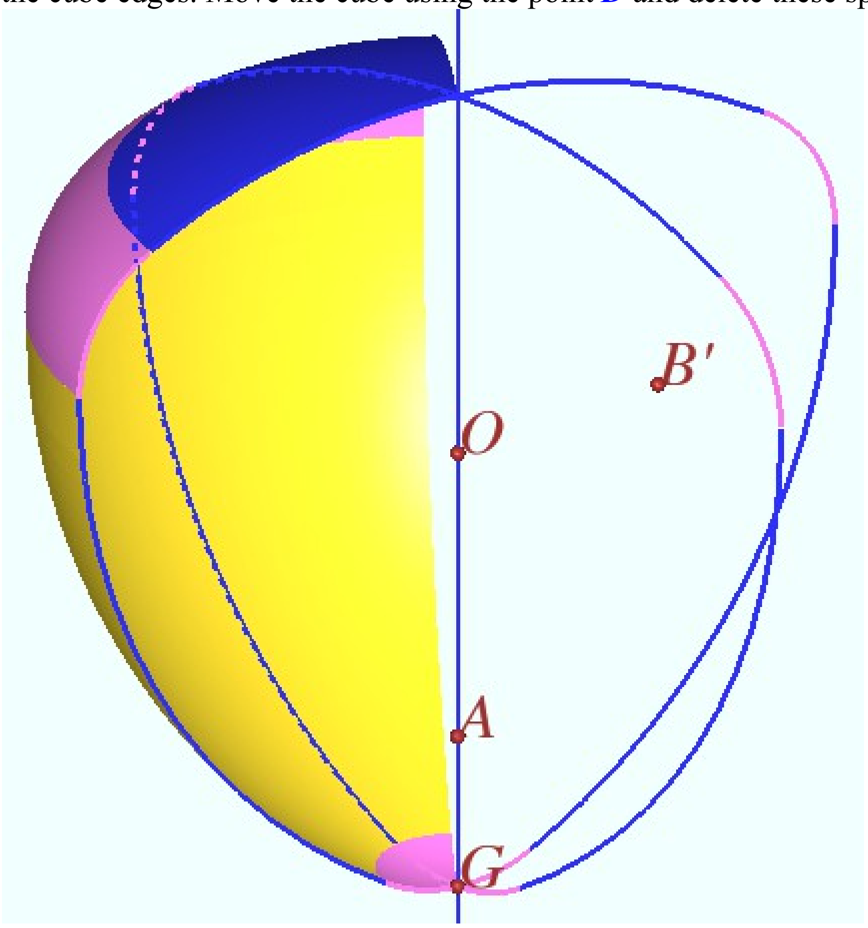

Figure 5. Surface of revolution of a Reuleaux triangle analogue construction

2.3 Surface of revolution of a Reuleaux pentagon

The original curve is a Reuleaux pentagon. Its construction is as follows: we take a regular pentagon with the side ***a*** and the diagonal ***h***, then replace the sides by the arcs centered at the opposite vertices and radii equal to the diagonal ***h***. The constructed curve is the Reuleaux pentagon. The shape consists of five arcs and is a curve of constant width ***h***.

In the Figure 6 the Reuleaux pentagon is determined by the center ***O*** and the vertex ***A*** from the reference regular pentagon.

The surface of revolution of a Reuleaux pentagon we obtain by Reuleaux pentagon rotating about one of its axis of symmetry, for example the symmetry axis ***OA***. The surface of revolution of a Reuleaux pentagon through one of its symmetry axes forms a surface of constant width.

Figure 6 shows the surface of revolution of a Reuleaux pentagon.

The point ***B'*** controls the rotation.

Make sure that the solid has a constant width.

Study may be performed with use of the cube ***DEF***. Set the angular position of the cube using the points ***E*** and ***F***. Then move the cube along the edges using the point ***D***. Examining the cube faces, we find the protruding elements of the solid as colored spots on the cube edges. Move the cube using the point ***D*** and delete these spots.

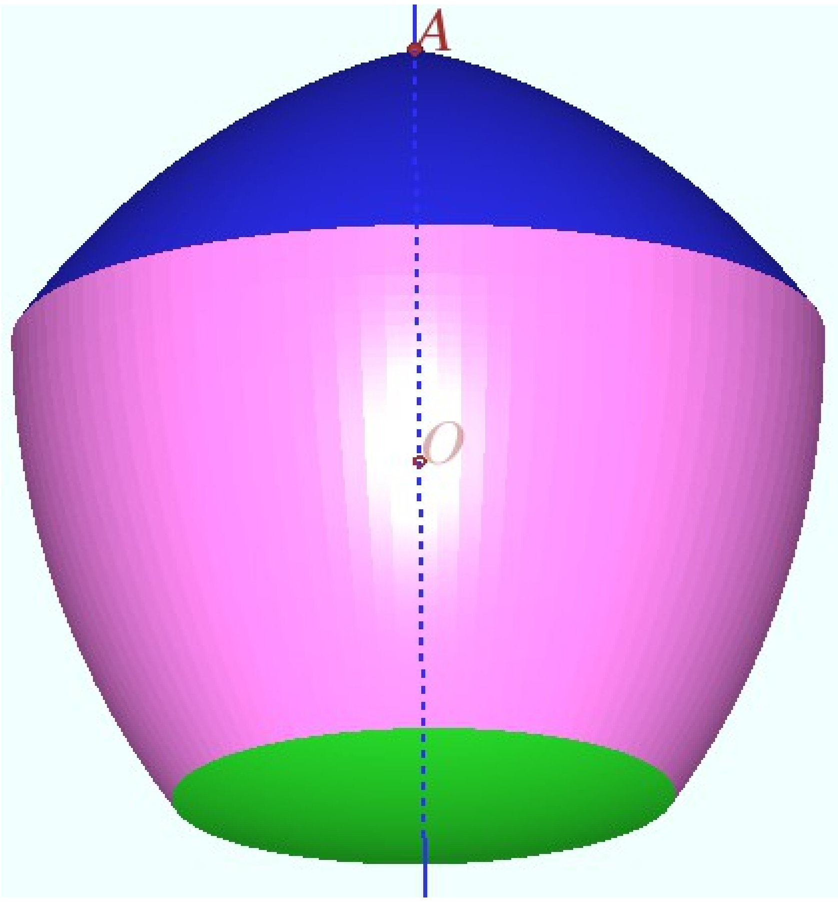

Figure 6. Surface of revolution of a Reuleaux pentagon

2.4 Meissner solid

Meissner created a solid of constant width on principles similar to which a Reuleaux triangle had been created. A Reuleaux triangle is constructed on the basis of the equilateral triangle with the side ***h*** using equal circles centered at the triangle vertices. We construct a Meissner solid on the base of the equilateral tetrahedron with the edge ***h*** by the same idea. A plain regular triangle is replaced by a spatial regular tetrahedron. Similar to replacing the segments by the arcs in the Reuleaux triangle, we replace the plain faces by the pieces of spheres centered at the tetrahedron vertices. We call them «sails».

Let's take the equilateral tetrahedron ***ABCD.*** We construct the pieces of spheres centered at the vertices ***A***, ***B*** and ***C*** with radii equal to the tetrahedron edge, bounded by triples of planes. One of them is the plane ***ABC***. Two other planes for each sphere contain the tetrahedron height ***DO*** and one each of the points ***A***, ***B*** and ***C***, which do not coincide with the center of the corresponding sphere. For example, the arc ***AD*** lying on the plane ***ADO*** is the curve of intersection of the spheres centered at the points ***B*** and ***C***. So the point ***F*** is the midpoint of the segment ***BC*** and the center of the circle containing the arc ***AD***. Three sails close the half-space under the plane ***ABC***. The distance from any point of the sail to the opposite vertex of the triangle ***ABC*** is equal to ***AB***.

We close the face ***ABC*** by such sail inside the trihedral angle ***DABC***. The distance from any point of the sail to the vertex ***D*** is also equal to ***AB***.

The sails are built, but three "gaps"remain between them. For example, the gap bounded by the arcs ***BC***. Meissner's idea is as follows: for each point ***E*** lying on the arc ***AD*** we create a circular arc ***BC***, all points of which are equidistant from ***E***. In Figure 7 the active point ***E*** may be moved through the arc ***AD***. The arc ***BC*** centered at ***E*** with the radius ***h*** and the ends ***B*** and ***C*** is constructed for each position of the point ***E***. As you move the point ***E*** the arc rotates without changing the shape. We replace a set of arcs by the curved patch, a piece of the surface of rotation of a circular arc. Three of these patches together with four sails form the tetrahedron of constant width or Meissner solid. The constructed solid passes into itself when rotated by 120° about the tetrahedron symmetry axis passing through the vertex ***D***. In Figure 7 three gaps form a triangle and should be replaced by the curved patches, the pieces of the surface of rotation of a circular arc.

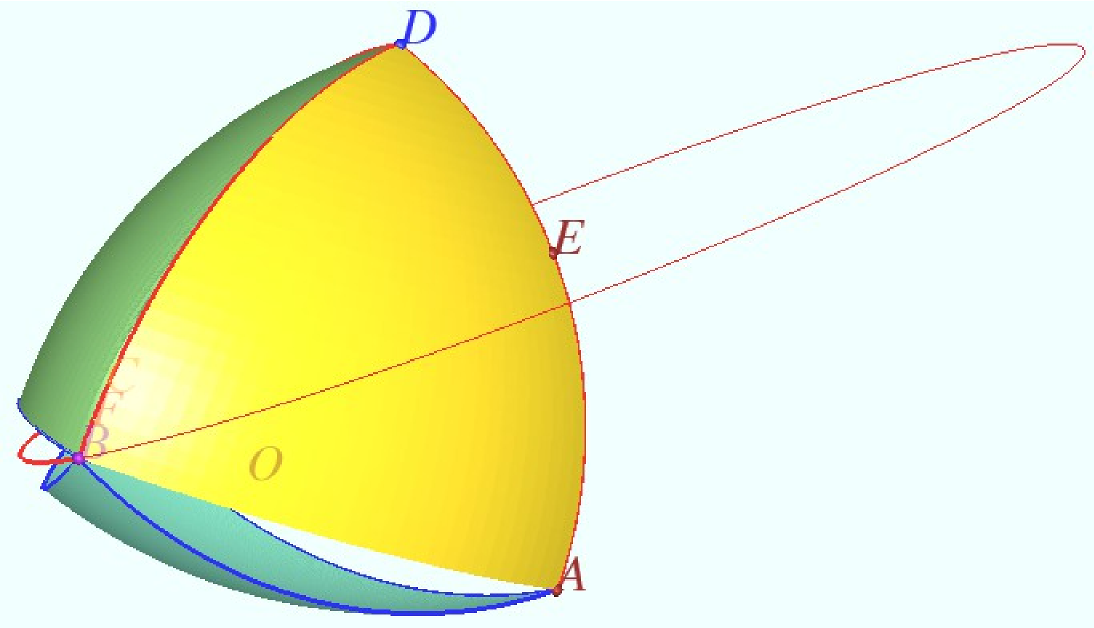

Figure 7. Meissner solid construction

2.5 Second Meissner solid

The second variant of the Meissner solid construction is the following.

Let's take the regular tetrahedron ***ABCD.*** The sail ***ABD*** is the piece of the sphere centered at the vertex ***C*** bounded by the planes of the faces ***ABC, BCD*** and the plane of symmetry ***ADF***. The point ***F*** is the midpoint of ***BC***.

For each point ***E*** lying on the arc ***AD*** we create a circular arc ***BC***, all points of which are equidistant from ***E*** (Meissner's idea). Let's move the point ***E***, watch along ***BC*** and along ***AD***. We replace the constructed set of arcs by the curved patch, a piece of the surface of rotation. It is shown in Figure 8 in red. Move the point ***E*** and check that the red surface contains any of the arcs ***BC***.

In Figure 8 the second Meissner solid is shown, which is one of two noncongruent shapes. In this Meissner solid three gaps, replaced by red curved patches, have a common vertex ***B***. They pass into one another when when rotated by 120° about the axis ***BM*** (***M*** is the centroid of ***ABCD***). The sails ***ACB*** and ***CDB*** we also obtain when rotating ***ABD*** by 120° about the tetrahedron symmetry axis passing through the vertex ***B***.

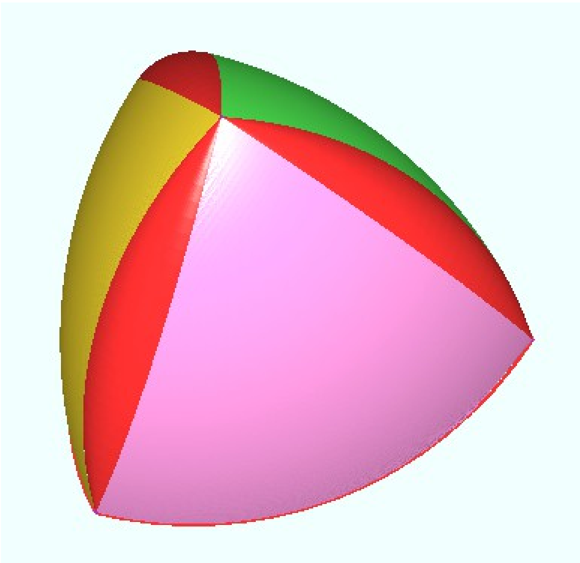

Figure 8. Second Meissner solid

2.6 The algebraic solid of revolution

The surface of revolution of an algebraic curve of constant width we obtain by algebraic curve rotating about one of its axis of symmetry, for example the symmetry axis ***OA***. The surface of revolution of an algebraic curve forms a surface of constant width.

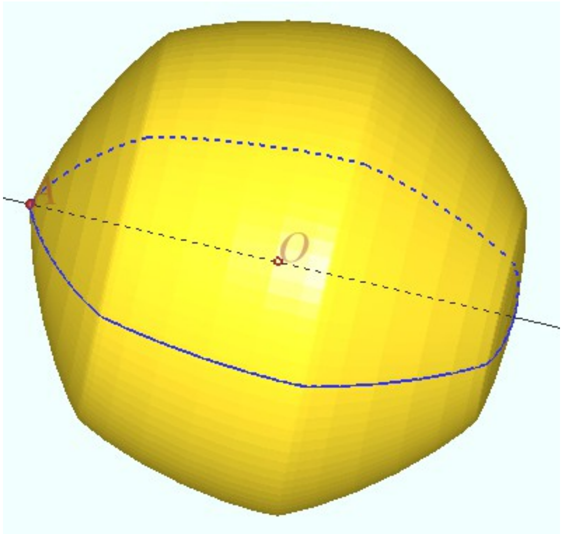

Figure 9. Surface of revolution of a constant width algebraic curve

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