Evolute and involute (evolvent)

Interactive solutions to problems s of differential geometry


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Materials of the interactive set can be used in Undergraduate education for the study of the foundations of differential geometry, it can be used for individual study methods of differential geometry. All figures are interactive if you install GlnMA program from http://www.deoma-cmd.ru/

Conception of Evolute

In the differential geometry of curves, the evolute of a curve is the locus of all its centers of curvature. Equivalently, it is the envelope of the normals to a curve. Apollonius (c. 200 BC) discussed evolutes in Book V of his Conics. However, Huygens is sometimes credited with being the first to study them (1673).

Equation of evolute

Let γ be a plane curve containing points \( \vec{X} = (x, y) \). The unit normal vector to the curve is \( \vec{n} \). The curvature of γ is \( K \). The center of curvature is the center of the osculating circle. It lies on the normal line through γ at a distance of \( \frac{1}{K} \) from γ in the direction determined by the sign of \( K \). In symbols, the center of curvature lies at the point \( \vec{z} = (\xi, \eta) \). As \( \vec{X} \) varies, the center of curvature traces out a plane curve, the evolute of γ and evolute equation is:

\[
\vec{z} = \vec{X} + \frac{\vec{n}}{K}.
\]  

(1)

Case 1 Let γ is given a general parameterization, say \( \vec{X} = (x(t), y(t)) \). Then the parametric equation of the evolute can be expressed in terms of the curvature \( K = \frac{y''}{(x'^2 + y'^2)^{3/2}} \). The unit normal vector to the curve is \( \vec{n} = \left( -y', x' \right) \sqrt{x'^2 + y'^2} \). Evolute equation is:

\[
\vec{z} = \left( x - \frac{y'(x'^2 + y'^2)}{x'y'' - x''y'}, y + \frac{x'(x'^2 + y'^2)}{x'y'' - x''y'} \right).
\]  

(2)

Case 2 Let γ is given an equation \( y = f(x) \). Then the curvature \( K = \frac{y'}{(1 + y'^2)^{3/2}} \), the unit normal vector to the curve is \( \vec{n} = \left( -y', 1 \right) \sqrt{1 + y'^2} \). Evolute equation is:

\[
\vec{z} = \left( x - \frac{y'(1 + y'^2)}{y''}, y + \frac{1 + y'^2}{y''} \right).
\]  

(3)

Case 3 Let γ is given an equation \( f(x, y) = 0 \). Then the curvature \( K = \frac{2f_x f_y f_{xx} - f_x^2 f_{xy} - f_y^2 f_{yy}}{(f_x^2 + f_y^2)^{3/2}} \). The unit normal vector to the curve is \( \vec{n} = \left( -f_x, f_y \right) \sqrt{f_x^2 + f_y^2} \). Evolute equation is:

\[
\vec{z} = \left( x - \frac{f_x \left(f_x^2 + f_y^2\right)}{2f_x f_y f_{xy} - f_x^2 f_{yy} - f_y^2 f_{xx}}, y + \frac{f_y \left(f_x^2 + f_y^2\right)}{2f_x f_y f_{xy} - f_x^2 f_{yy} - f_y^2 f_{xx}} \right).
\]  

(4)

Case 4 Let γ is given an equation \( R = f(s) \), where \( s \) — длина дуги, отсчитываемая от некоторой точки. Let evolute equation is: \( \vec{R} = f(\vec{z}) \). Обозначим углы, составляемые касательными с
ось абсцисс \( \alpha \) и \( \tilde{\alpha} \). Then \[
\frac{d\alpha}{d\bar{s}} = \frac{1}{\bar{R}}, \quad \frac{d\tilde{\alpha}}{d\tilde{\bar{s}}} = \frac{1}{\bar{R}}, \quad d\bar{s} = dR, \quad \alpha = \tilde{\alpha} + \frac{\pi}{2}, \quad d\alpha = d\tilde{\alpha}.
\]
Evolute equation is:
\[
\bar{R} = R \frac{dR}{d\bar{s}}, \quad \bar{s} = R + c.
\]

Typical figure and its using
In each figure points 0 and 1 on the x-axis specify the Cartesian coordinates. The graph of the original curve is shown by the blue line. It is determined by the parameters which values are indicated in the figure. The parameters are set by the active points. The evolute is shown in red. The trial points labeled \( C, D, E, \ldots \) are located on the curve. Tangent circle and its center point \( C', D', \ldots \) belonging to evolute are shown in pink.

The process of learning and investigation one can begin from the verification of the basic property of the evolute. By moving point \( C \), make sure that the center of curvature belongs to the evolute at all positions of point \( C \). By activating the "Properties" button find and check the equations used for the curves construction. Explore several evolute curves. Save the file in a convenient location and change the original curve. Perform calculations and construct evolute of the selected curve. Build a tangent circle to check your construction.

Evolutes samples

The evolute of the parabola

The curve given by the equation \( y = f(x) = k x^2 \). Then the curvature of the curve is \( K = \frac{y'''}{(1+y'^2)^{3/2}} = \frac{2k}{(1+(2 k x^2)^2)^{3/2}} \), the normal is \( \tilde{n} = \frac{(-y',1)}{\sqrt{1+y'^2}} = \frac{(-2 k x, 1)}{\sqrt{1+(2 k x)^2}} \).

The equation of the evolute is:
\[
\tilde{\xi} = \left( x - \frac{y' (1+y'^2)}{y''}, y + \frac{1+y'^2}{y''} \right) = \left( x - x (1+4 k^2 x^2), k x^2 + \frac{1+4 k^2 x^2}{2 k} \right) = \left( -4 k^2 x^2, 3 k x^2 + \frac{1}{2 k} \right).
\]

The evolute of the ellipse

The curve given by the equation \( f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \). Then the equation of the evolute is:
\[
\tilde{\xi} = \left( x - \frac{f_x (f_x^2 + f_y^2)}{2 f_x f_y f_{xy} - f_x^2 f_{yy} - f_y^2 f_{xx}}, y + \frac{f_y (f_x^2 + f_y^2)}{2 f_x f_y f_{xy} - f_x^2 f_{yy} - f_y^2 f_{xx}} \right) = \left( \frac{x^3}{a^4}, -\frac{y^3}{b^4} \right) (a^2 - b^2).
\]

Fig.1. Evolute of the parabola  
Fig.2. Evolute of the ellipse
The evolute of a hyperbola

The curve given by the equation \( f(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \). We perform the parameterization \( x = a \cosh t, y = b \sinh t \). Then the equation of the evolute is:

\[
\vec{\zeta} = \left( x - \frac{y'}{x'y'' - x''y'}, y + \frac{x'}{x'y'' - x''y'} \right) = \left( \frac{\cosh^3 t}{a}, \frac{-\sinh^3 t}{b} \right) (a^2 + b^2) = \left( \frac{x^3}{a^2}, \frac{-y^3}{b^2} \right) (a^2 + b^2).
\]

The evolute of an astroid

The curve given by the equation \( \{x, y\} = \{a \cos^3 t, a \sin^3 t\} \). The curvature of the curve is \( K = \frac{2}{3a \sin 2t} \). Then the equation of the evolute is:

\[
\vec{\zeta} = a \left( \cos t (3 - 2 \cos^2 t), \sin t (1 + 2 \cos^2 t) \right).
\]

Involute

The involute of the plane curve is a curve with respect to which the given curve is the evolute. The involute is a curve, for which the normal at each point is tangent to the initial curve. That is, the involute contains the tangents to the given curve. Imagine that the little pebble lies on the curve at an arbitrary point \( P \). Let the thread run from the pebble along the curve. Then the involute determines the trajectory of pebble, moving away from the curve and associated with it by the stretched thread. They say that the selected point moves along the involute when unwinding the thread, which lies on the curve. The family of the involutes generated by different points of the surface exists for each curve. Further we consider only one of them, starting at the origin.

Case 1. Let this curve be given parametrically by the expression \( y = f(x) \). Then the unit tangent vector is \( \vec{\tau} = \left( 1, \frac{y'}{\sqrt{1 + (y')^2}} \right) \). The length of the thread unwound from the beginning is \( s = \int_0^1 \sqrt{1 + (y')^2} \, dx \). The equation of the involute \( \vec{\zeta} = (\xi, \eta) \) has the form of:

\[
\vec{\zeta} = \vec{X} - s \vec{\tau} = \begin{pmatrix} \int_0^x \sqrt{1 + (y')^2} \, dx \\ y \int_0^x \frac{1}{\sqrt{1 + (y')^2}} \, dx \end{pmatrix} = \begin{pmatrix} \int_0^x \sqrt{1 + (y')^2} \, dx \\ y \int_0^x \frac{1}{\sqrt{1 + (y')^2}} \, dx \end{pmatrix}.
\]
Case 2. Let the curve be given parametrically by the expression \( \mathbf{X}(t) = (x(t), y(t)) \). Then the unit tangent vector is \( \mathbf{\tau} = \frac{(x'(t), y'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}} \). The length of the thread unwound from the beginning is 
\[
s = \int_0^t \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2} \, dt.
\]
The equation of the involute \( \mathbf{\zeta}(s) = (\xi(s), \eta(s)) \) has the form of:

\[
\mathbf{\zeta} = \mathbf{X} - s \mathbf{\tau} = \left( x - \frac{x' \int_0^t \sqrt{x'^2 + y'^2} \, dt}{\sqrt{x'^2 + y'^2}}, y - \frac{y' \int_0^t \sqrt{x'^2 + y'^2} \, dt}{\sqrt{x'^2 + y'^2}} \right).
\] (7)

Case 3. Let the curve be given by the expression \( \mathbf{R}(s) = f(\tilde{s}) \). Using (5), we find the equation of the involute:

\[
\mathbf{R} = \mathbf{\tilde{R}} \quad \text{and} \quad s = \int \frac{\tilde{s} \, d\tilde{s}}{R}.
\] (8)

Typical figure and it's using

In each figure points 0 and 1 on the x-axis specify the Cartesian coordinates. The graph of the original curve is shown by the blue line. It is determined by the parameters which values are indicated in the figure. The parameters are set by the active points. The involute is shown in red. The trial points labeled \( C, D, E, \ldots \) are located on the curve. Tangent circle and its center point \( C', D', \ldots \) belonging to involute are shown in pink.

The process of learning and investigation one can begin from the verification of the basic property of the involute. By moving point \( C \), make sure that the segment \( CC'(DD', \ldots) \) is tangent to the given original curve at all positions of point \( C \), and point \( C(D, E, \ldots) \) is the center of curvature of the involute. By activating the "Properties" button find and check the equations used for the curves construction. Explore several involute curves. Save the file in a convenient location and change the original curve. Perform calculations and construct involute of the selected curve. Build a tangent circle to check your construction.

Involutes samples

The involute of the parabola

The curve given by the equation \( y = kx^2 \). Then the unit tangent vector is \( \mathbf{\tau} = \frac{(1, 2kx)}{\sqrt{1+4k^2x^2}} \). The length of the thread unwound from the beginning is:

\[
s = \int_0^t \sqrt{1+4k^2x^2} \, dx = \frac{x}{2}\sqrt{1+4k^2x^2} + \frac{1}{4k} \ln\left(2kx + \sqrt{1+4k^2x^2}\right).
\]

The equation of the involute has the form of:

\[
\mathbf{\zeta} = \left( x - \frac{s}{\sqrt{1+4k^2x^2}}, y - \frac{2kxs}{\sqrt{1+4k^2x^2}} \right) = \left( \frac{x}{2} - \frac{\arcsinh(2kx)}{4k\sqrt{1+4k^2x^2}}, \frac{x - \arcsinh(2kx)}{2\sqrt{1+4k^2x^2}} \right).
\]
The involute of the circle

The curve given by the equation \( \vec{R} = a \). We use (8) and obtain:

\[
R = \bar{s}, \quad s = \int \bar{s} \, d\bar{s} = \frac{\bar{s}^2}{2a}.
\]

The equation of the involute has the form of: \( R^2 = 2as \). The curve in parametrically form:

\[
\vec{\zeta} = a (\cos t + t \sin t, \sin t - t \cos t).
\]

The involute of the astroid

The curve given by the equation \( \vec{X} = (a \cos^3 t, a \sin^3 t) \). Then the unit tangent vector is \( \vec{t} = (-\cos t, \sin t) \). The length of the thread unwound from the beginning is: \( s = 1.5a \sin 2t \). The equation of the involute has the form of:

\[
\vec{\zeta} = \frac{a}{2} (\cos t (3 - \cos^2 t), -\sin^2 t).
\]

Literature