Advanced Geometry

Interactive textbook

1 Flexible polyhedrons

This interactive textbook is destined for students who don't like formulae, but experience great interest in images and their dynamic transformation. The chapter presents the topic "Flexible polyhedrons". Dynamic GInMA illustrations provide maximum interactivity, gives the opportunity to work with 3D-images and make it convenient to introduce the study in training courses on geometry.

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Install the software and enjoy working with interactive drawings by clicking on the Figures in the text.

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1.1 A flexible quadrangular mesh after Hellmuth STACHEL [1]

Let's construct a flexible quadrangular mesh according to Stachel (Stachel [1]) which consists of equal quadrilaterals and may form a tiling on a cylinder.

A quadrangular flexible mesh consists of congruent convex quadrangles seen as rigid bodies and only dihedral angles along internal edges can vary. Let a polygonal mesh be extracted from the planar tessellation in such a way, that with any three faces with a common vertex also the fourth face through this vertex is included. This quadrangular mesh is continuously flexible if and only if the initial quadrangle is convex.

**Step 1.** For the mesh constructing we define the base quadrangle $ABCD$. The side $AB$ is defined by the points $A$ and $O$ ($O$ is the midpoint of $AB$). The $BC$ length is defined by the coefficient $k = \frac{BC}{AB}$. The angles at the vertices $A$, $B$ and $C$ are equal to $\alpha$, $\beta$ and $\gamma$, respectively.

**Step 2.** Let the angle between the planes $ABC$ and $ACD$ be equal to $\phi$. We construct the quadrangle $BACD_1$ equal to $ABCD$.

**Step 3.** We construct the point $E$ according to the conditions:

a) $BE = CD$,

b) $\angle BCD = \angle DAB = \delta = 2\pi - (\alpha + \beta + \gamma)$,

c) $\angle EBC = \angle BCD = \gamma$.

The pink circle is the locus for which the conditions a) and b) are fulfilled.

**Step 4.** The black circle is the locus for which the conditions a) and c) are fulfilled. The vertices $G$ and $G'$ coincide with the points of intersection of the circles.

**Steps 5,6.** The forth vertices of the quadrangles $D_1A_1GB$ and $B_1CGQ$, each equal to $ABCD$ are constructed. A flexion of a $2 \times 2$ tessellation mesh is constructed. We fulfill the further construction using the iterated half-turns. We see, that the initial data may result in two different meshes.

**Check** the Stachel statement: This quadrangular mesh is continuously flexible if and only if the initial quadrangle is convex.

**Step 7.** Let's define the midpoints of the segments $BC$, $BD_1$, $BG'$ as $X$, $Y$, $Z$. Let's construct the bisectors of dihedral angles using these points and the point $O$ (black lines) and find the common perpendicular $HH'$ to them (shown by red). These bisectors are the half-turns axes.

**Check** the Stachel statement: The axes of the four half-turns have a common perpendicular $HH'$.

**Steps 8-21.** Let $F$ be the midpoint of $AD$. Let us construct the perpendicular from $F$ to $HH'$. This perpendicular is the axis of a half-turn mapping the quadrangle $ABCD$ into the mesh element $DA_2C_2A$.

The next mesh elements are consistently produced by such half-turns.

The images of two neighboring quadrangles under iterated coaxial helical displacement cover the complete mesh. We construct the mesh element $A_2B_2C_2D_2$ using helical displacement of $ABCD$ (displacement at $2F'H'$ and rotation at twice the angle between the $FF'$ and $HH'$).

At each nontrivial flexion obtainable by a self-motion the vertices are located on a cylinder of revolution. All vertices of the flexion arise from $A$, $B$, $C$ or $D$ by displacements which keep the common perpendicular $HH'$ of the half-turns’ axes fixed. This is the reason why all vertices have the same distance to $HH'$. Three helix curves containing a set of mesh vertices are shown. Each of them contain the point $B$ and neighbor vertices. Use the point $E$ to make the cylinder.
1.2 Flexible chain of solids according to Jen–chung Chuan

Let's construct a group solids which may rotate as a whole. Let this group consists of the solids with the given pair of opposite edges. This group was constructed by Jen–chung Chuan in [2].

**Step 1.** We define a pair of half-planes which intersect at the angle $E_1EE_1'$ equal to a predetermined value $\alpha < 90^\circ$. If $\alpha = \frac{2\pi}{n}$, $n = 3, 4, \ldots$, then the half-planes form a group of $n$ equal dihedral angles, each of which passes into the nearby angle under the symmetry with respect to the half-plane which separates them.

We take the point $O'$ on the halfplane $EE_1'E_0$ and construct its projection $O$ on the other half-plane $EE_2E_0$. We rotate the straight line $PO'$ using either the point $P$, or the angle $\phi$ between $PO'$ and the edge of the dihedral angle $EE_0$. We define the point of intersection of $PO'$ and the edge $E_0E$ as the point $F$.

**Step 2.** We construct the points $K$ and $K'$ on the sides of the angle $O'FO$ by such that $FK' = FO$, $FK = FO'$. The triangle $\Delta FK'K = \Delta FOO'$ is constructed. As $O'O \perp FK$, then $KK' \perp FO'$ and $KK' = OO'$. Hence, for the fixed point $O'$ the segment $KK'$ has a constant length.

**Step 3.** Let the point $A$ lies on $FK'$, and $AK' = \frac{KK'}{\sqrt{2}}$, point $D$ lies on the perpendicular to the plane $FK'K$, and $DK = AK'$. Then $AK'$ lies on the plane $EE_1'E_0$, $DK$ on the plane $EE_2E_0$, and the polygonal line $AK'KD$ is the polygonal line connecting two vertices of a regular tetrahedron and containing its midline.

**Step 4.** Let the points $B$ and $C$ are symmetric to the points $A$ and $D$ with respect to $K'$ and $K$, respectively. Then $ABCD$ is a regular tetrahedron. Its size and the position on the interactive figure are defined by the points $E$ and $O'$ and the angles $\alpha$ and $\phi$ (or the point $P$).

**Step 5.** The regular tetrahedron $ABCD$ is constructed. The edge $CD$ lies on the plane $EE_1E_0$, the edge $AB$ lies on the plane $EE_1'E_0$. Check it out, looking along the edge $EE_0$.

**Step 6.** We construct a tetrahedra symmetrically about the planes of the dihedral angle. The tetrahedron $A_1B_1C_1D$ is symmetric to $ABCD$ with respect to the plane $ECE_0$. The tetrahedron $ABC_2D_2$ is symmetric to $ABCD$ with respect to the plane $EAE_0$.

**Шаг 7–10.** Each tetrahedron is inscribed in the dihedral angle with the same value $\alpha$, therefore all reflections occur in the planes, the angles between which are the same. So the straight line $E_0E$ is common. The following tetrahedra consistently are produced by reflecting:

$A_2B_2C_2D_2$ is symmetric to $ABC_2D_2$ with respect to the plane $E_0EC_2$.

$A_3B_3C_3D_3$ is symmetric to $A_1B_1C_1D_1$ with respect to the plane $E_0E_1A_1$.

$A_2B_2C_3D_3$ is symmetric to $A_2B_2C_2D_2$ with respect to the plane $E_0EA_2$.

$A_3B_3C_3D_3$ is symmetric to $A_2B_2C_3D_3$ with respect to the plane $E_0EC_3$.

When we define $\alpha = 45^\circ$, then the points coincide $A_3 = A_4$ and $B_3 = B_4$, obtained from each other by turning on $360^\circ$. The chain of eight tetrahedra closes.

**Шаг 11.** When we define $\alpha = 36^\circ$, and construct two more tetrahedra (the tetrahedron $A_4B_4C_4D_4$ is symmetric to $A_2B_2C_1D_1$ with respect to the plane $E_0EA_4$...), then the points coincide $C_4 = C_4'$ and $D_4 = D_4'$. The chain of ten tetrahedra closes.

1.3 Connelly flexible polyhedron

We construct the Connelly flexible polyhedron. The segment $AD = a = 12$ defines the scale. $AC = a$, $CD = d = 11$. The position of the moving point $B$ is determined by the distances $AB = b = 10$, $BC = c = 5$. The point $O$ is determined by the distances $OB = a$, $OC = b$, $OD = b$. The point $E$ is determined by the distances $EO = a$, $EA = b$, $ED = c$. The point $Q$ is determined by the distances $QO = l = 17$, $QE = QB = a$. The point $C'$ is determined by the distances $C'A = a$, $C'Q = b$, $C'B = c$. The point $D'$ is determined by the distances $D'A = a$, $D'Q = b$, $D'B = c$. $C'D' = d$. The Connelly flexible polyhedron is constructed. Use $B$-point and find the conditions for self-intersection.

2 Moving elements in space

2.1 Cardan Joint

Research. The Cardan Joint transmits the rotation from the shaft $OA$ to the shaft $OA'$. Two shafts shown schematically by the rays from the points $A$ and $A'$ are mounted at an angle $\beta$ to one another. Corresponding straight lines intersect at the point $A$. One of the shafts, for example $OA$, rotates the outer drive, while the other is rotating under the action of the first shaft. In a vehicle the first shaft rotate an engine. The second shaft is connected with the wheels. The shafts are ended by mounting clamps, shown in the form of semicircles $BA'C$ and $B'A'C'$. The axis are inserted into the clamps rotatable. They are shown in the Figure in the form of the segments $BC \perp OA$ and $B'C' \perp OA'$. These axes are structurally related, and they are perpendicular to each other forming the cross $BC \perp B'C'$. The equations that describe the structure are as follows:

$$\vec{OA} = (0, 0, 1), \quad \vec{OA'} = (-\sin \beta, 0, -\cos \beta),$$

$$\vec{OB} = (\cos \alpha, \sin \alpha, 0), \quad \vec{OB'} = (-\cos \beta \sin \gamma, \cos \gamma, \sin \beta \sin \gamma), \quad \text{where} \quad \tan \gamma \cos \beta = \tan \alpha.$$

Check that the vectors are really mutually perpendicular.

![Figure 6. Cardan Joint](image)

2.2 Sliced solids (tetrahedron and prism)

**Problem.** Cut an arbitrary tetrahedron by a plane into two parts so that it was possible to form the same tetrahedron by folded parts otherwise. [1, 193].

**Solution.** Let the angles are $\angle DEG = \angle BDC$, $\angle DEF = \angle ADC$. Then $\angle ADEF = \angle AEDG$, $\angle ADFG = \angle AEGF$. We cut the tetrahedron by the plane $EFG$ and turn tetrahedron $DEFG$, interchanging $D$ and $E$, $F$ and $G$. Let $O$ and $Q$ be the midpoints of $DE$ and $FG$. Use button $\phi$ and rotate $DEFG$. Lock it in any position using the Stop button. The rotation interchanges $D$ and $E$, $F$ and $G$. The condition is satisfied. Interactive drawing contains rotatable element which may be locked in any position using the Stop button.

**Problem.** Cut a triangular prism into three equidimensional tetrahedra.

![Figure 7. Sliced tetrahedron and prism](image)

2.3 Quartet of pairwise touching triangles

**Problem.** Place four nonoverlapping triangles on the plane so that any two triangles touched by the part of its sides having a non-zero length.

**Advice.** Suppose that one of the triangles is regular, the other triangles are equal to each other and arranged symmetrically …

![Figure 8. Quartet of pairwise touching triangles](image)

2.4 Eight pairwise touching tetrahedra

**Problem.** Place eight nonoverlapping tetrahedra so that any two tetrahedra touched by the part of their faces having a nonzero area [2, 389].

**Solution.** We use the symmetry. We construct four touching tetrahedra standing on the plane $ABC$ by using four triangles lying in the common base plane of tetrahedra. We use four triangles touched by the part of their sides having a non-zero length on the plane of the base $ABC$. We construct four pyramids above the plane $ABC$ with bases equal to the triangles found in the previous problem. Let $D$ be the common vertex for all tetrachedra. Then we construct four pyramids under the plane $ABC$ with bases equal to the described above triangles.

**Research.** Rotate the set of base triangles of upper tetrachedra with respect to the triangles of lower tetrachedra using $A'$. Make sure that the base of each of the upper tetrachedra and the base of each of lower tetrachedra may overlap simultaneously.

![Figure 9. Eight pairwise touching tetrahedra](image)
2.5 Three regular tetrahedra inside a cube

**Problem.** Place inside a cube three regular tetrahedra having no common interior points with edges equal to the edge of the cube.

**Research.** Each of tetrahedra contains the central point $P$, which provides translational movement of the tetrahedron. Two of the vertices control the rotation. Points $E, G$ and $I$ provide a turn in either direction. Points $F, H$ and $J$ provide only rotation around the axes $EP, GP$ and $IP$, respectively. Try to move any tetrahedron inside the cube. Further make the task more complicate, superpose the edge of the tetrahedron with the selected cube edge. And finally superpose the edge of the tetrahedron with the selected cube edge and at the same time the midpoint of the opposite tetrahedron edge superpose with the cube center.

**Solution.** Notice that the distance between the opposite tetrahedron edges is exactly equal to the distance from the center of the cube to its edges. Therefore tetrahedra may be placed between three mutually perpendicular cube edges and cube center. In this case the midpoint of one tetrahedron edge coincide with the cube center. All tetrahedra touch in the cube center. The right image in Figure shows a possible arrangement of tetrahedra.

Figure 10. Three regular tetrahedra in a cube
2.6 Locus of the vertex of a tetrahedron with a fixed base

**Problem.** Find the locus of the vertex $D$ of the tetrahedron $ABCD$. The base $ABC$ and the segment $MN$ length are given. $MN$ connects the midpoints of the edges $BC$ and $AD$.

**Research.** Choose the base $ABC$ and $MN$ length. *Explore*, where may be placed the vertex $D$.

**Solution.** Let us consider the point $O$, symmetric to $A$ with respect to $M$. Note that $OD = 2MN$. Hence the locus of the vertex $D$ of tetrahedron $ABCD$ is a sphere with a radius $2MN$ centered at the point symmetrical to $A$ with respect to $M$.

![Figure 11. Locus of the tetrahedron vertex](image)

2.7 Circle tangent to quadrant faces and parallel to the quadrant edge

**Problem.** The circle of radius $R$ touches three faces of the quadrant $OXYZ$, and its plane is parallel to one of the edges (for example $OZ$). Find the locus of points of tangency of the circle with the faces and the locus of the circle centers.

**Research.** The position and the size of the configuration is determined by the points $O$ and $X$. *Explore* the behavior of the circle when you move the point $A$ (the plane of moving is parallel to $OZ$). Notice how the points of tangency of the circle with faces move.

**Solution.** The circle center and the third point of tangency of the circle with the plane $OXY$ move by parallel arcs of a circle with radius $R$. This is the locus of the midpoints of the segment (diameter) which ends slide along mutually perpendicular segments. We find the locus as an aggregate of three similar sets. If the circle lies in the angle face, it touches the face by all its points. We find the center of the circle $I$ using the conditions $AI \perp OXY \Rightarrow AI \parallel OZ$, $|AI| = R$. The points of tangency are $B$ and $C$ ($BC \parallel OXY$). The diameter is $BC|OXY$, $|BC| = 2R$. The point $B$ moves along a segment with the length $2R$ parallel to $OX$. The midpoints of the segments having the length $2R$ move along the arcs of radius $R$ centered at $OZ$. We show the locus of the circles centers in green, the points of tangency in blue.

![Figure 12. Circle tangent to quadrant faces and parallel to the quadrant edge](image)

**2.8 Circle tangent to all quadrant faces**

**Problem.** The circle of radius $R$ touches three faces of the quadrant $OXYZ$. Find the locus of the circle centers.

**Research.** The position and the size of the configuration is determined by the points $O$ and $Z$. **Explore** the behavior of the circle when you move the point $A$.

**Solution.** Let the angle of inclination of the circle plane to the plane $OXY$ is equal to $\alpha$, the angle of inclination of the circle plane to the plane $OYZ$ is equal to $\beta$, the angle of inclination of the circle plane to the plane $OXZ$ is equal to $\gamma$. It is known that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Let $B$ be the point of tangency, $AB = R$, $AH \perp OXY$. Then $\angle ABH = \alpha$, $AH = R \sin \alpha$. The vector is $OA = R \{\sin \beta, \sin \gamma, \sin \alpha\}$. Note that

$$AO^2 = R^2(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = R^2(3 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma) = 2R^2.$$

Therefore, the point $A$ is located on a sphere with center $O$ and radius, and not more than a distance $R$ from the faces. In general case, we may construct four circles of radius $R$, which have a center at the point $A$ and touch the angle faces. The circle center sets a pair of possible points of tangency with each face, and we may arbitrarily choose the points of tangency on two faces. The point of tangency of the circle with the third face is fixed.

![Figure 13. Circle tangent to all quadrant faces](image)
3 References


