Why should We Use Visualization within Math Science and Math Education?

Vladimir Shelomovskii
e–mail: vvsss@rambler.ru
Murmansk State University
Russia

Abstract

Visualization is powerful and effective tool for understanding and in–depth study of mathematics. It helps students to better understand the foundations of mathematics, engineers to calculate complicated geometric constructions, and researchers to find new mathematical regularities. Modern mathematical science presents such complicated events and phenomena, that even the talented scientists often pass by hidden nuances, unobvious mathematical rules, effects and properties in their math research. We show some examples of scientific research of ATCM 2011 and ICGG–2012 authors, based on math visualization provided by GInMA software. We also show the examples of the calculations which have became possible largely due to the fact that the authors were able to observe with GInMA the computational grid behavior, and have compared the observed image and arising calculation difficulties. Visualization is absolutely necessary in math education. Modern students often memorize standard rule for solving a problem and do not understand meaning of the statement standing behind the rule. We demonstrate the examples of the solutions and their visualization using GInMA, which helps to understand meaning of mathematical statements. With the use of GInMA software we construct derived surfaces and curves, pedal curves, caustic curves and surfaces, complicated intersecting solids, such as the solid formed by intersection of an ellipsoid and a sphere, and many other nonstandard objects, which represent the set of interrelated geometric objects.

All the pictures in the electronic version of this paper are interactive. Install GInMA software from the website
http://deoma–cmd.ru/en/Products/Geometry/GInMA.aspx
click on the Figures and investigate interactive solutions of the problems.

1. Introduction

Mathematics attends on every human life, regardless of profession and education. These are practical numeracy, elementary analysis and logic in everyday life, complicated formulas in the calculations of engineers, and abstract math rules and discoveries of scientists–mathematicians, which are even difficult for understanding of the human mind. But we should not think that this science is isolated from the rest of human life. If the study of mathematics is not simply the memorizing of standard rules for solutions of some problems, but the understanding of meaning which stands behind the rules, then it gives more than just the computing skills. It forms the analytical, logical thinking, the ability to formulate the problem, regardless of whether it is a mathematical task or the task set before us by life itself, and forms the ability to think through tactics and strategy of the solution. From this point of view, visualization is very important, as it is powerful and effective tool of understanding and in–depth study of mathematics.

We live in the world full of modern technologies. It demands from the education to form people with common vision of situation concerning technology, within their chosen field of activity, in order to either create something new (new constructions, new technology) or to operate effectively in unusual situations (the pilot in the aircraft with failed equipment, the military in the situation with the deficiency of resources, and so on). We may visually and emotionally see the difference between the standard approach to the situation with technological difficulties and the

reasonable approach of the cogitative person in a series of films "The investigation of aircraft accidents" posted on YouTube. For example, there are shown the simple failures of aircraft devices, such as the fuel level sensor failure during the flight of Tun Inter over the Mediterranean 06.08.2005 [14], the pressure sensor failure on the Air France during the flight over the Atlantic Ocean 01.06.2009 [15], and so on. Diligent implementation of standard test sheets during mentioned incidents led to the deaths of the pilots and the hundreds of passengers seated behind them. Success in emergency situation is associated with the vision of the situation as a whole. Such example is landing of the aircraft on the Hudson 15.01.2009 [16]. The pilot of this flight was not doing standard actions, he immediately turned on the emergency source of electricity, and categorically rejected the variants of returning at the airport of departure, proposed by his dispatchers. He spent available time to prepare for landing on water, as a result all the people in the falling aircraft survived. The investigative modeling of the situation showed that the attempts to implement standard variants of behavior in the accident most likely led to death.

When good teacher teaches geometry, he forms the skill to think the problem through to clarity, to see the situation as a whole. In the old days, this benefit was evident, as the older generation was surrounded by the "mechanical" technique with a high proportion of geometricity. Homo sapiens understood the machinery he used and the principles of its work, and could operate successfully in a failure of technology. The current generation of students live in a magical world, where users shouldn't know the structure and the principles of operation of their electronic assistants, however being sure, that they will perform any task by some unknown (magic) way. In this world the habit to think the problem through to clarity is in contradiction with the typical forms of thinking. Modern students more and more often learn standard solutions of problems and usually doesn't understand meaning of rules standing behind the statements. For example, suppose we know that the equation $a x^2 + b x + c = 0$ has the roots $x_1 = 3, x_2 = 13$, and $x_3 = 2013$. We are needed to find the coefficient b. If you don't understand the situation, it seems that the problem has no solution. If you understand, the solution is trivial. Visualization helps us to understand the mathematical statements, to think the problem through to clarity, to see the situation as a whole.

In this paper we show the benefit of visualization in various fields, at all levels of mathematics: school, university, technical, scientific. We provide the examples of visualization of various mathematical solutions with the use of GInMA software [GInMA]. Interactive GInMA files have been developed aiming to enable clear seeing the configuration in space, to provide changing of the construction and viewing from different angles. This approach helps students to better understand the foundations of mathematics, engineers to calculate complicated geometric constructions, researchers to find new mathematical regularities.

All the pictures in the electronic version of this paper are interactive. Install GInMA software from the website
http://deoma–cmd.ru/en/Products/Geometry/GInMA.aspx
click on the Figures and investigate interactive solutions of problems.

2. Imagination Problem

2.1 I don't imagine

People often say the words: "I don't imagine ...". Desire to imagine something is the basis of many developments. Visual presentation makes it easy to understand mathematical statements. For example, students of pedagogical faculty of Moscow State University (Russia's leading university)

could not imagine without having interactive GInMA illustration, how one can drill a hole in a cube in such a way, that through this cube one could hold another cube equal to the given one (Fig.2.1). The other problem, which forced students to cry: "It is impossible!", was the problem about eight mouses living in the touching houses. Each house has the form of a tetrahedron and they are located in such a way, that each mouse has the holes in it's tetrahedron home allowing it to reach any of its seven mouse–friends in their tetrahedron houses. This is the problem of eight tetrahedra which are tangent in pairs by non–zero areas of its surfaces. Students also solved it using GInMA (Fig.2.2). As well, with the help of GInMA software students imagine how eight various spheres can touch all of the four planes of tetrahedron faces (Fig. 2.3).
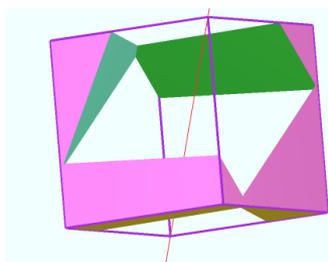
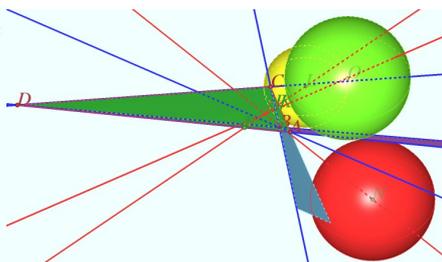**Figure 2.1** Hole in a cube   **Figure 2.2** Eight tetrahedra touching in pairs   **Figure 2.3** Eight spheres touching all the planes of tetrahedron faces
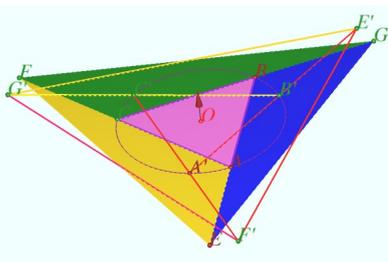
2.2 Math Visualization at school

To show students math objects and phenomena, which are hard to imagine, teachers use a computer and appropriate mathematical software, CAS and DGS. Some of teachers are trying to create their own visual materials for math lessons. But not all the teachers are enthusiastic in using the computer in their classrooms. Experience of Russian education shows that the most of math software is made by the computer specialists according to the standards of computer technicians, which have a little to do with that is customary for teachers. The result teacher's reaction is the suspension of computer use, especially by the older generation of teachers. A teacher has no idea why should he spend time and make efforts to study the alien logic of the computer program and to learn programming. In our opinion, the best way is mass use of ready–made electronic teaching materials, made by the experts, may be by the most enthusiastic, educated math teachers, easily communicating with computer and software. Then for the ordinary teacher, who is not ready to spend time for developing his own support materials for math lesson, it is very convenient to enter interactive file through the textbook Figure by single click on the image. This idea is implemented in GInMA software.

3. Scientific Research with GInMA

Scientific research in the field of mathematics require extensive and deep knowledge, special talent of thinking and imagination. Many talented mathematicians can easily imagine abstract mathematical regularities, which are inaccessible for understanding and imagination of an ordinary man, far from the mathematical sciences. But modern mathematical science presents such complicated events and phenomena, that even the talented scientists often pass by hidden nuances,

unobvious mathematical rules, effects and properties in their math research. We show some examples of small scientific research based on math visualization, provided by GInMA software.

3.1 Elliptical cross sections of the quadric surface

Problem 1. [1] Let a quadratic surface and a set of the parallel planes be given. Let point T be a point at which the plane touches the quadratic surface. We have one such point for a paraboloid, we have two such points for an ellipsoid and a hyperboloid of two sheets, and points $\vec{T}'$ and $\vec{T}$ are symmetrical about the origin. We make the following observations

1. A set the of the cross sections of the given quadric by parallel planes is a family of the similar ellipses.

2. We say that:

– a diameter of the paraboloid is a beam, parallel to its axis and beginning at point $\vec{T}$;

– a diameter of the ellipsoid is a segment passing through its center and connecting points $\vec{T}'$ and $\vec{T}$;

– a diameter of the hyperboloid of two sheets is a beam, passing through its center and outgoing into the infinity from points $\vec{T}'$ and $\vec{T}$ and lying on the straight line $\vec{T}\vec{T}'$.

The diameter contains the centers of the ellipses from the ellipses family obtained by the intersection of the quadric and the family of planes parallel to the given plane, which touches the quadric at point T . Figures below illustrate the statements 1 and 2. They show the solids cut off by the plane from the paraboloid (Fig. 3.1), the ellipsoid (Fig. 3.2) and the hyperboloid of two sheets (Fig.3.3). By varying the parameter m_0, we construct the family of parallel planes and watch as the center of the elliptical cross–section slides in the diameter.
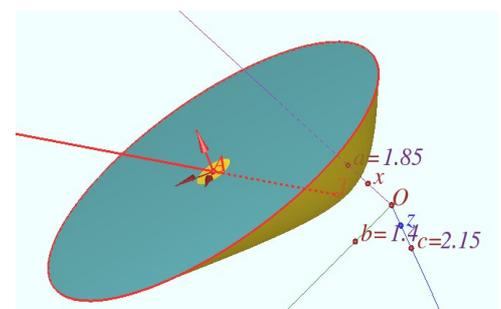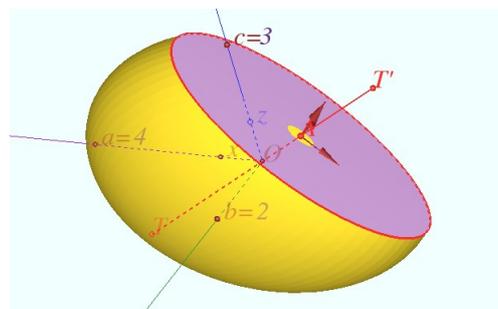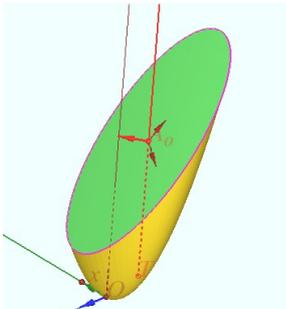

Figure 3.1 Paraboloid cut off by the plane   **Figure 3.2** Ellipsoid cut off by the plane   **Figure 3.3** Hyperboloid cut off by the plane

3.2 Pavillet tetrahedron

The Pavillet tetrahedron have been found by Axel Pavillet (Canada) [2]. It is simple construction which gives students great opportunity of in–depth study of perpendicularity in space and testing of various methods of proof.

Problem 2. [3] Let the base triangle ABC be given. Let ABC incircle be centered at I point and A_1, B_1, C_1 be points where the incircle touches the base triangle sides BC, AC, AB, respectively. Let the segments AA', BB', CC' be perpendicular to ABC plane, where $AA'=AB_1, BB'=BA_1, CC'=CA_1$. Let us name the solid $IA'B'C'$ as the Pavillet's tetrahedron for the triangle ABC.

Figure 3.4 shows two properties of the Pavillet tetrahedron. The proof of the set of properties have been derived using GInMA.
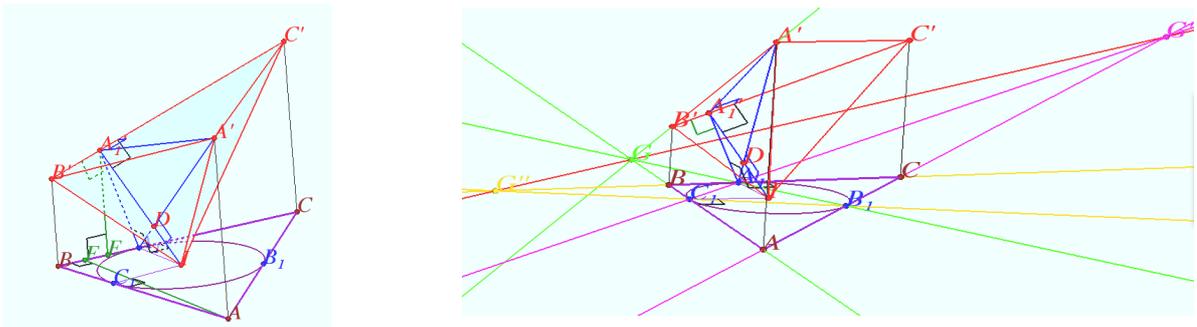
Figure 3.4 Properties of Pavillet tetrahedron

3.3 Perfect circles and amicable triangles

Michel Sejfried (Poland) have found some new properties of perfect circles and amicable triangles [4]. The issue was in constructing the required configuration. GInMA software helped to construct the configuration and to prove some difficult theorems using the properties of space transformations. The problem was formulated as following:

Problem 3. Let triangle ABC be given, point A_0 lies on BC, B_0 lies on AC, C_0 lies on AB, AA_0 intersect BB_0 at F, AA_0 intersect CC_0 at H, BB_0 intersect CC_0 at K. Find point A_0 such that points $F, H,$ and K lie on the incircle of ABC.

We have foud the transformation of a plane converting an arbitrary triangle into an equilateral triangle, and then we used the properties of projective geometry, see Figure 3.5.
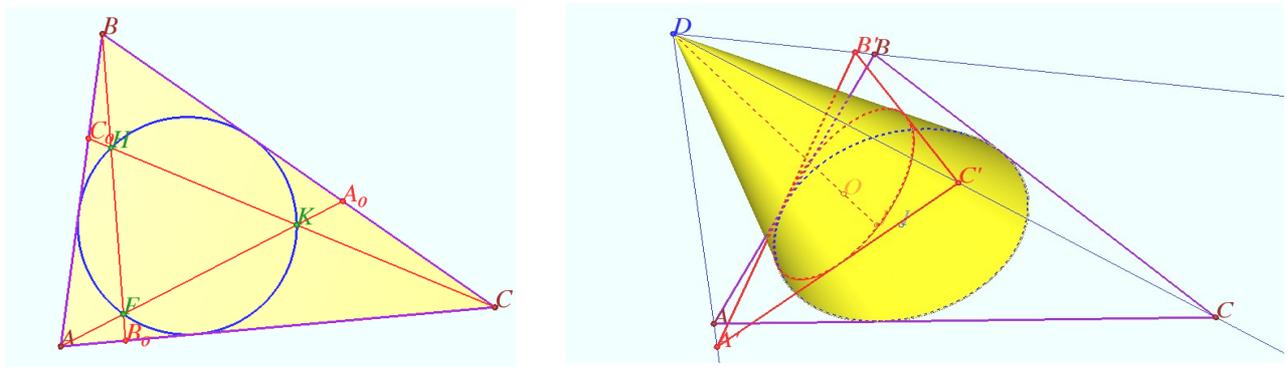

Figure 3.5 Construction of Sejfried configuration

3.4 Mirroring problem

A situation similar to the previous one have arose in finding the shortest path inside a regular tetrahedron. We performed mirror reflections of the tetrahedron with GInMA, rotated configuration and saw something that turned into a statement.

Problem 4. Prove that the regular tetrahedron and the sequence of its mirroring images are inscribed in a cylinder with the axis crossing the medians of the faces and dividing them at a ratio $3:2$ from the vertex.

Fragment of the solution: Distances from any vertex to the axis are equal to $0.3\,a\,\sqrt{3}$, where a is the edge of the tetrahedron.

We have built and explored the construction using GInMA, see Figure 3.6.
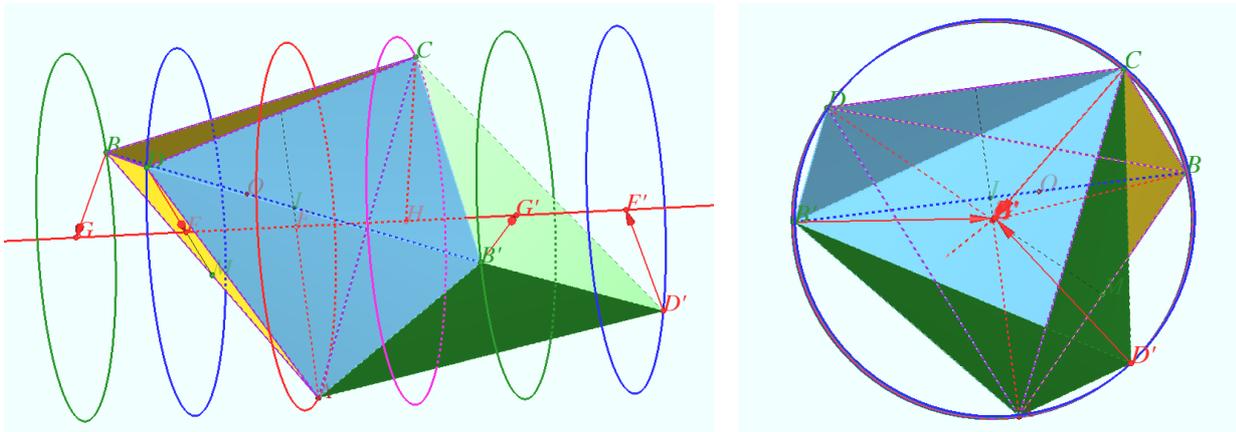
Figure 3.6 Regular tetrahedron and the sequence of its mirroring images inscribed in a cylinder

4. Technical Developments with GInMA

In design, architecture and other applied sciences the configurations are often applied which are created with the use of some standard object as a basis, and then the superpositions of many such objects are made, possibly with the change of its size, as in fractals. The properties of these configurations may be quite unusual, and the possibility of visualization is very important.

You may see Helical tubes touching one another, which were made by Mayrhofer and Walter from University of Innsbruck (Austria) [5] on Figure 4.1. You may see flexible quadrangulars tiling a cylinder described by Stachel from Vienna University of Technology (Austria) [6] on Figure 4.2. We can easily change the parameters and observe the obtained constructions.
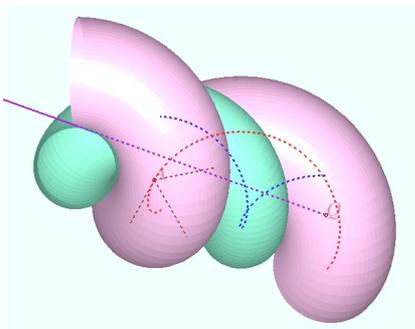

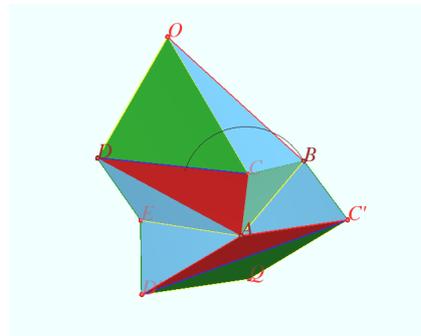**Figure 4.1** Helical tubes   **Figure 4.2** Flexible polyhedron

5. Calculus Mathematics

Complicated calculations, particularly the calculations of spatial configurations, make it necessary to perform a set of transformations, to select the roots after solving the equations, and to understand the results. In school mathematics, similar issues arise in solving the problems with parameters. Many of the problems with parameters admit geometric interpretation. Visualization of geometric interpretation allows students to understand the meaning of formal conversion.

5.1 I know how to calculate, but I want to see

Geometry has a lot of sections, formal explanation of which is not promising. The examples are derivative surfaces and curves [7]. There are formulas which define the transformations of the objects. For example, the equations are known that allow us to construct pedal or orthotomic surface for given surface, or a caustics for given curve. We are always curious to see, is it really, that the characteristic relations are satisfied for any point of the curve, say, the perpendiculars always fall into the required point. GInMA software allows us to see it clearly, Figures 5.1 – 5.5.
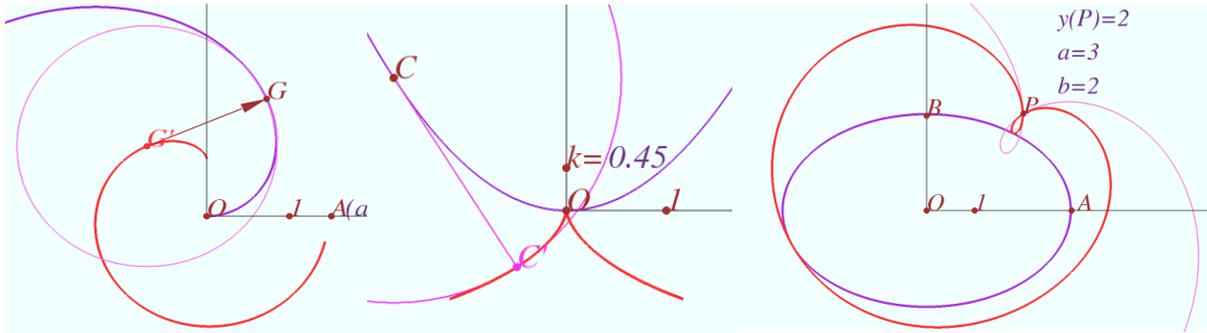

Figure 5.1 Evolute for the curve   **Figure 5.2** Evolvent for the curve   **Figure 5.3** Pedal curve and orthotomics
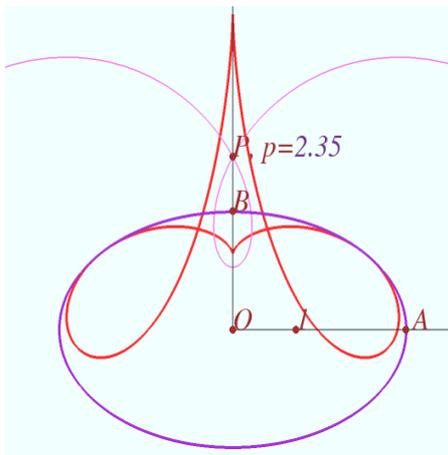

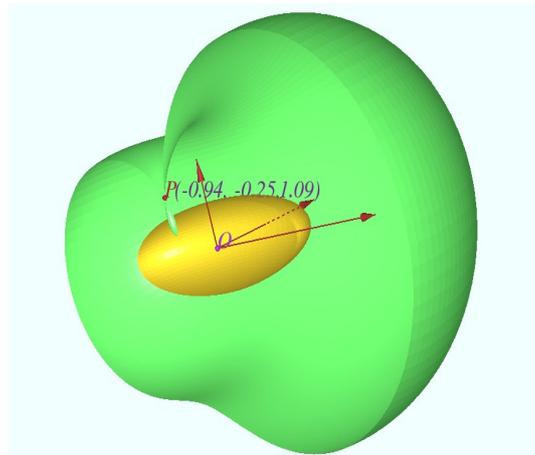**Figure 5.4** Caustic curve   **Figure 5.5** Pedal surface and orthotomics

5.2 Is there any solution?

The proofs sometimes begin with the implicit assumptions. For example, we have the definition of Dandelin spheres and folowing statements [8]: "The inner and outer spheres tangent internally to a cone and also to a plane intersecting the cone are called Dandelin spheres. The spheres can be used to show that the intersection of the plane with the cone is an ellipse. Let be a plane intersecting a right circular cone with vertex in the curve…" But do such spheres exist for arbitrary ellipse? How many solutions has the problem? Using GInMA, we can easily carry out the construction, which gives the answer to both questions, see Figure 5.6.
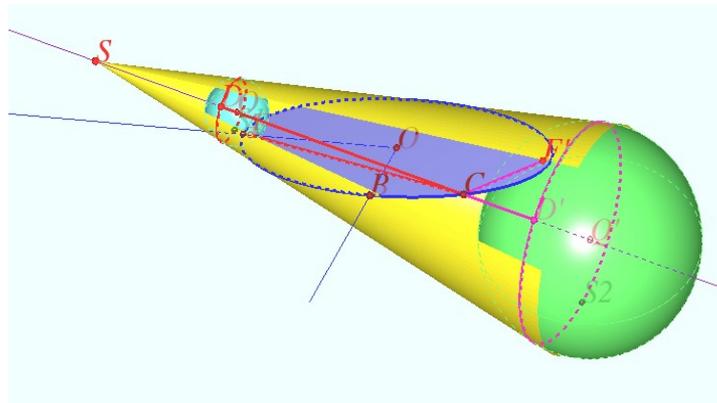

Figure 5.6 Dandelin spheres for arbitrary ellipse

5.3 Are the transformations performed correctly?

When the mathematician is performing complicated transformations, it is important to check the validity of the transformations. Math programs help not enough, they can't really simplify the obtained expressions. The verification may be fulfilled as visual checking. Let as see the following example [9]: the curve of intersection of a sphere and an ellipsoid shown on Figure 5.7.
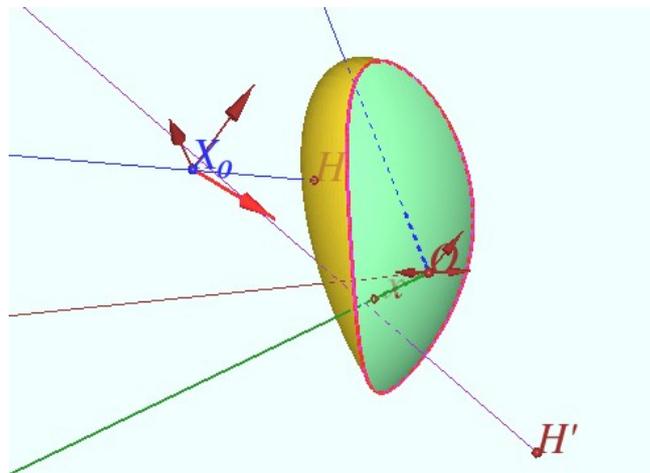

Figure 5.7. Solid of intersection of the sphere and the ellipsoid.

5.4 Which solution should we choose?

A similar problem is choosing the required solution among many possible ones. This problem is nontrivial, for example, when we are finding the volume or the surface area of intersecting solids, when the intersection curve may form a loops. Here, the opportunity to see such a curve is very useful. In order to find a way to calculate the integrals with the required accuracy over the surface bounded by complicated, possibly self–intersecting curve, it is necessary to find adequate model and to explore the possibility of independent control.

The author took part in the research of Professor Wei–Chi Yang and Professor Sylvester Tompson concerning finding the surface area and the volume of the solid formed by intersection of an ellipsoid and a sphere. It needs the computation of the integrals over the surface bounded by the line of intersection which is quite complicated curve, see Figure 5.8. The calculations became possible largely due to the fact that the researchers were able to observe with GInMA the computational grid behavior with the skips of the solutions, and had compared the observed image with arising calculation difficulties.
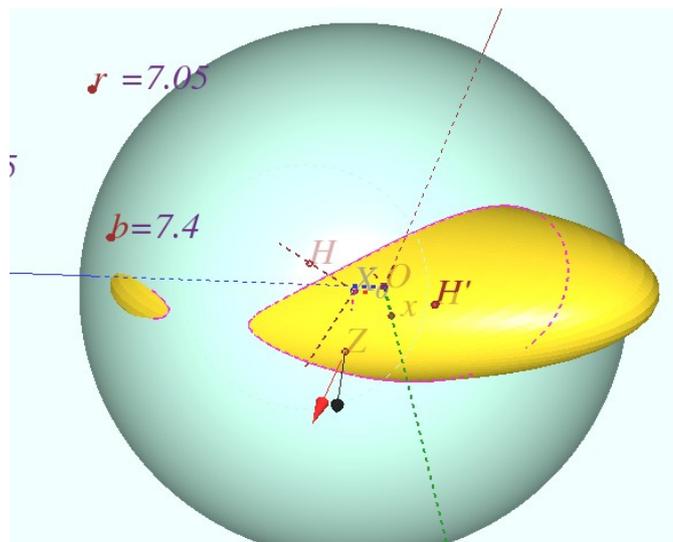

Figure 5.8. Curve of intersection of the sphere and the ellipsoid.

5.5 What is the essence of the solution?

Let us consider the following problem [11].

Problem. Let $f(x,y,a)=1-\sqrt{x^2-2axy+y^2-4y+5}$. Suppose the expression gets its maximum value only for single pair of (x,y). Find the range of the parameter $a\in[-1,1]$.

Formal answer.is the following: $a\in\left[-\dfrac{1}{\sqrt{5}};\dfrac{1}{\sqrt{5}}\right]$. Seeing the image of the surface given by the

equation on Figures 5.9 and 5.10, students may understand, why there is bifurcation of the solutions.
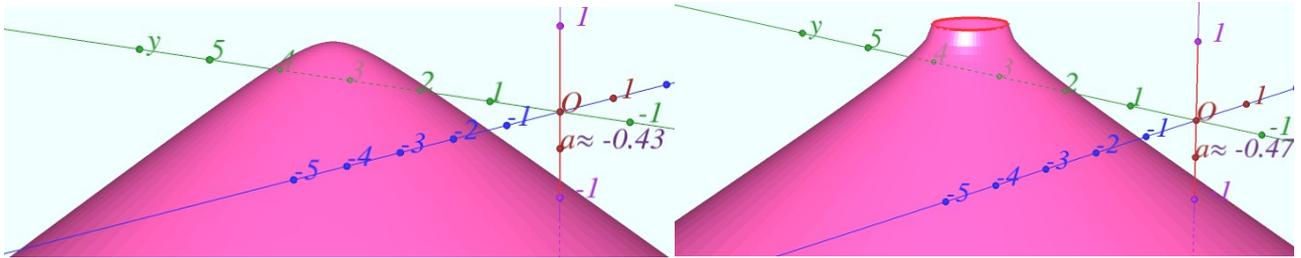

Figure 5.9. Solution for $a \in \left[-\dfrac{1}{\sqrt{5}}; \dfrac{1}{\sqrt{5}} \right]$. **Figure 5.10.** Solution for $|a| > \dfrac{1}{\sqrt{5}}$.

6. Community of various branches of mathematics

Mathematics is one of few sciences where you can observe the relationship between seemingly very different parts. So, there are Riemann (spherical) geometry and Hyperbolic geometry besides Euclidean geometry. These three sections of geometry are different in basic postulates. At the same time for solving the problem with the use of the Hyperbolic geometry model it is convenient to use the constructions and communications, found in Riemannian geometry [11], [12], see Figures 6.1 and 6.2.
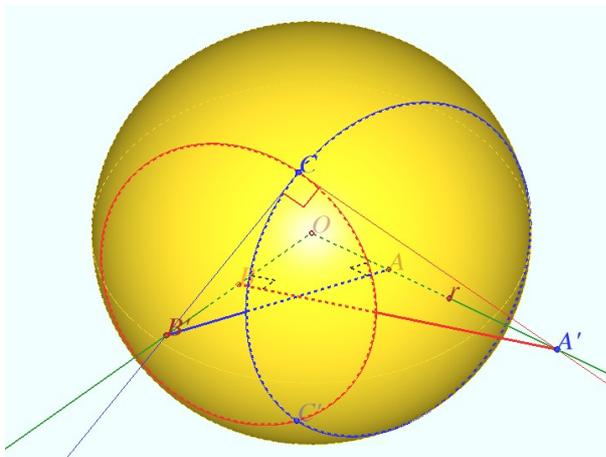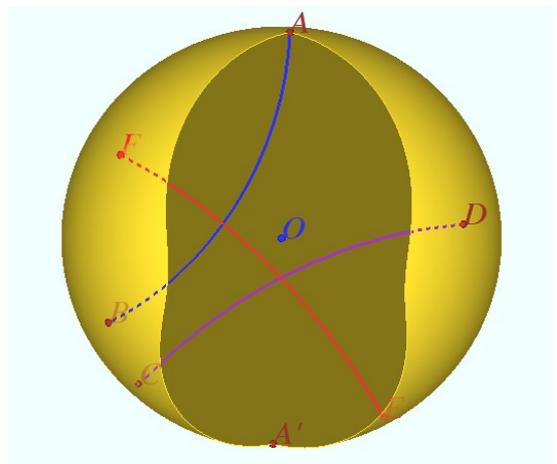

Figure 6.1. Perpendicularity in Riemannian geometry

Figure 6.2. Solution using perpendicularity in the model of Hyperbolic geometry

7. Attractive mathematics

Math may be interesting, funny, attractive. With the use of GInMA software we demonstrate the objects, which are really the sets of interrelated standard geometric solids and curves. For example,

it may be the Mathematical Fish [13], constructed as the combination of an ellipsoid, two spheres, and four surfaces, see Figure 7.1.
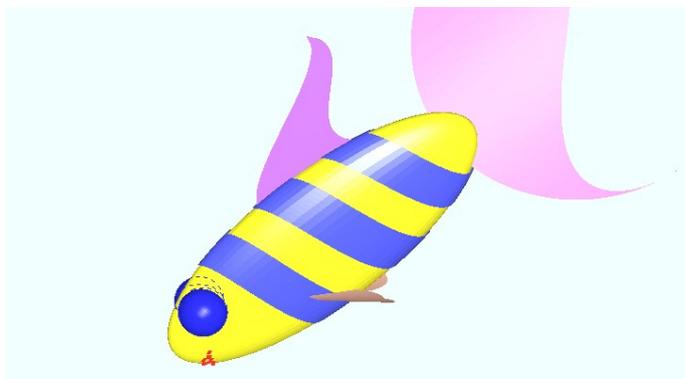

Figure 7.1. Mathematical Fish

8. Conclusion

1. Interactive visualization in mathematics greatly extends the capabilities of both teaching and research.
2. Experience in conducting research with interactive GInMA illustrations shows the effectiveness of GInMA use in the following cases:
– for student's understanding of complicated geometrical concepts (caustics, orthomiks, and so on),
– for creating of interactive teaching materials in geometry and algebra in the form of text documents: pdf, doc, odt, html, and so on.
– for monitoring the correctness of the calculations when we perform transformations, the result of which can be visualized.

9. References

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[15] Crash of Air France flight 447 accident investigation. http://youtu.be/xDsy–TCLfwg

[16] Air Crash Investigation – Hudson River Runway US Airways Flight 1549. http://youtu.be/IbTMmqw2_XI

Software Packages

[GInMA]   GInMA, 2012, S. Nosulya, D. Shelomovskii, and V. Shelomovskii. http://deoma–cmd.ru/en/Products/Geometry/GInMA.aspx